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Superfluidity in the AdS/CFT correspondence

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ABSTRACT: A chemical potential may be introduced into the AdS/CFT correspondence by setting the D3 branes of the construction spinning. In the field theory the fermionic modes are expected to condense as Cooper pairs, although at zero temperature the chemical potential destabilizes the scalar sector of the $\mathcal{N} = 4$ theory obscuring this phenomena. We show, in the case where a chemical potential is introduced for a small number of the gauge colours, that there is a metastable vacuum for the scalar fields where fermionic Cooper pairing is apparently manifest. In this vacuum the D3 branes expand non-commutatively (to balance the centrifugal force) into a D5 brane, in a mechanism analogous to Harmark and Savvidy's (M)atrix theory construction of a spinning D2 brane. We show that the D5 brane acts as a source for the RR 3-form whose UV scaling and symmetries are those of a fermion bilinear. The D5 brane rotates within the S^5 and so decays by the emission of RR fields which we interpret as the metastable vacuum decaying via higher dimension operators.

KEYWORDS: D-branes, AdS-CFT Correspondance.

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Contents

1. Introduction

The AdS/CFT correspondence [1], which is a duality between a four dimensional strongly coupled gauge theory and a weakly coupled gravity background (anti-de-Sitter space) in one higher dimension, provides a new, controlled, framework in which to describe the properties of strongly interacting systems. The original duality for the conformal $\mathcal{N}=4$ super Yang-Mills theory at the origin of moduli space resulted from consideration of the limit where the gauge theory on the surface of N coincident D3 branes decoupled from the supergravity description of the bulk spacetime. Many aspects of strong dynamics have since been explored by deforming the $\mathcal{N}=4$ theory by the inclusion of finite temperature [2] or relevant operators [3, 4, 5]. Motion in the radial direction of the AdS spacetime is interpreted as renormalization group flow in the field theory. The resulting theories display gravity descriptions of, amongst other properties, confinement, fermion condensates, instantons, and thermal phase transitions. In this paper we will address another familiar aspect of field theory, Cooper pair formation in a fermionic system at high density.

A number of authors [6, 11] have already studied the AdS/CFT correspondence at high density though they have concentrated on the scalar sector of the $\mathcal{N}=4$ gauge theory. The system may be placed at high density by the inclusion of a chemical potential. The chemical potential may be thought of as the vev of the temporal component of a spurious gauge field associated with a conserved global U(1) symmetry. It is natural to pick a U(1) subgroup of the global $\mathrm{SU}(4)_R$ group of the $\mathcal{N}=4$ theory. In the gravity dual the $\mathrm{SU}(4)_R$ symmetry appears as a gauged symmetry of the IIB supergravity on $AdS_5\times S^5$. As was recognized in [6] the gauge field originates in the 10 dimensional supergravity theory as an element of the metric and can be made non-zero by spinning the D3 branes of the construction on the S^5

space. This can be naively seen by starting with a static construction and performing a boost to a slowly rotating reference frame where an angular coordinate in some plane takes the form $\phi' = \phi + \omega t$. There are now $\phi' t$ components of the Lorentz transformation matrix Λ_{ij} and hence $\phi' t$ components in the metric.

The phenomena we wish to study is Cooper pair formation. High density fermionic systems have of course been much studied in condensed matter [7] but the relativistic analogues have also recently become an area of interest in QCD where a rich phase structure of colour superconductors has been uncovered [8]. In both the non-relativistic [7] and relativistic cases [8, 9] there are fairly rigorous arguments that *in the presence of a Fermi surface any attractive interaction gives rise to Cooper pair formation*. We will briefly review the renormalization group [7, 9] justification for this and the form of the condensate expected as a result of the gauge interactions in the $\mathcal{N} = 4$ theory. The preferred condensate is a colour singlet so the theory is a superfluid rather than a colour superconductor.

Capturing this phenomena in the $\mathcal{N} = 4$ SYM theory is complicated by the scalar sector of the theory which also transforms under the $\mathrm{SU}(4)_R$ symmetry. When the chemical potential is introduced as a spurious gauge field vev, scalar operators are also introduced that destabilize the moduli space of the theory. In the gravity dual this is clear; the D3 branes experience no potential on the six dimensional space transverse to their world volume (corresponding to the existence of the moduli space in the field theory) so there is no central force that can be used to sustain rotational motion (the source for the gauge field vev). Formally one should cure this problem by the inclusion of a positive scalar mass or, as previous authors have considered, by including finite temperature which indirectly generates such a mass.

We shall make do with finding a metastable vacuum in the scalar sector to avoid such complications. The D3 brane positions in the transverse space are described by the vevs of the adjoint scalar fields in the field theory on their surfaces and they can therefore be separated in a non-commutative fashion at the expense of energy of the form $\mathrm{Tr}\,[\varphi^\dagger, \varphi]^2$. We will use this central force to stabilize rotation of the D3 branes and provide us with the metastable vacuum. The construction is essentially that in (M)atrix theory of a spinning (fuzzy) D2 brane with surface D0 charge introduced by Harmark and Savvidy [10] and we will follow their methodology. Expanding branes non-commutatively induces couplings to higher dimensional RR forms as was realized by Myers [13]. The resulting configuration of expanded D3 branes may be thought of as a D5 brane with surface D3 brane charge. It would be nice to find supergravity backgrounds when all the D3 branes are spinning but in this paper we shall restrict ourselves to the case where only a small, probing, number of the D3 are spinning so the background geometry remains AdS. In the field theory this will imply that we have put a chemical potential in for a restricted set of gauge degrees of freedom. Nevertheless this will be sufficient to see the important physical effect of Cooper pair formation in this subsector of the theory.

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The D5 acts as a source for the RR 6-form or its dual description, a 2-form. This whole story is of course very analogous to the Polchinksi Strassler [4] analysis of the supergravity dual of the $\mathcal{N} = 1^*$ gauge theory. There, a mass that breaks $\mathcal{N} = 4$ to $\mathcal{N} = 1$ in the infra-red is introduced by placing the D3 brane construction in an appropriate background 3-form field strength. The 3-form provides a radial potential for the D3 branes with a minimum resulting from the interplay with the energy of the non-commutative expansion of the D3s. They find a fermion condensate as a sub-leading term in the 3-form solution. For our case this is the leading part of the 3-form. We show, following their methods, that the symmetries and RG scaling dimension of this induced form match with the formation of a Cooper pair condensate. In fact there is a triple scalar operator with the same R-charge and scaling dimension as the condensate which therefore mixes making the precise field theory interpretation difficult although the consistency of the story is appealing.

The D3/D5 construction we produce has explicit time dependence since the D5 topples in a six dimensional space and hence it spins down by loss of energy to RR-form and gravity waves. Harmark and Savvidy [10] studied these emissions extensively for the construction in Minkowski space and the same phenomena is apparent in AdS. Since these forms correspond to sources and vevs of field theory operators, the coupling of the world volume scalars to them may be thought of as higher dimension operators. These operators allow the metastable vacuum to decay to the true run-away vacuum and hence the angular momentum (R-charge) of the construction eventually resides on the asymptotic edge of moduli space.

A detailed study of the thermodynamics of the AdS/CFT correspondence at finite temperature and density was made in [11].[1] The absence of a superfluid phase was noted in [12] but the instability of our construction suggests that the phenomena will never be seen in the true vacuum of the $\mathcal{N} = 4$ theory.

2. The field theory

$\mathcal{N} = 4$ SYM theory may be placed at high density by the inclusion of a chemical potential, μ, for the gauginos

$$\Delta L = i\mu\lambda_i T_{ij}\gamma_0\lambda_j \,. \tag{2.1}$$

The chemical potential may be thought of as the vev of the temporal component of a spurious gauge field associated with a conserved global U(1) symmetry, with generator T_{ij}. In the AdS/CFT it is natural to pick a U(1) subgroup of the global $\mathrm{SU}(4)_R$ group of the $\mathcal{N} = 4$ theory and in this paper we shall restrict ourselves

[1] We are very grateful to Clifford Johnson for long discussion on this phase structure from which this work emerged and also for introducing us to spinning branes and their connection to the chemical potential.

to the fundamental representation generator $T_{ij} = \mathrm{diag}(1,1,1,-3)$. In the gravity dual the $\mathrm{SU}(4)_R$ symmetry appears as a gauged symmetry of the IIB supergravity on $AdS_5 \times S^5$. As discussed in the introduction the gauge field originates in the $10d$ supergravity theory as an element of the metric and can be made non-zero by spinning the D3 branes of the construction on the S^5 space.

In terms of SO(6), the isometry group of the S^5, the generator we study corresponds to an equal rotation in each of three SO(2) sub-groups/planes of the six dimensional space transverse to the D3 branes.

As a result of this motion we will create a D5 brane wrapped on a 2-sphere. The choice of different rotations in the three planes generates ellipsoidal configurations as discussed in [10] but for simplicity we will restrict to the most symmetric case.

Giving a vev to this temporal gauge field does not simply induce the term in (2.1) but also a scalar term coming from the (spurious) scalar covariant derivative

$$|D^\mu \varphi|^2 \rightarrow \sum_{i=1}^{6} g^2 \mu^2 |\varphi_i|^2 \,, \tag{2.2}$$

which is a positive quadratic term in the lagrangian and hence a negative mass term in the potential (here g is the gauge coupling). Since the $\mathcal{N}=4$ theory has a scalar moduli space this term serves to destabilize it and the theory is unbounded! The gravity dual of this phenomena may be seen by attempting to place a spinning D3 brane probe in the AdS background. The $AdS_5 \times S^5$ background is

$$ds^2 = H^{-1/2} dx_{//}^2 + H^{1/2}(dr^2 + r^2 d\Omega_5^2) \,, \qquad C_4 = H^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,, \tag{2.3}$$

where $x_{//}$ are the 3+1 dimensions of the D3 world volume and r and Ω_5 describe the six transverse dimensions. H is L^4/r^4 in AdS, with L the radius of AdS (in general H can be a more complicated harmonic function in which case the background describes the $\mathcal{N}=4$ theory at a generic point in the moduli space). Note that under a dilatation in the four dimensional x^μ space the radial coordinate r transforms with mass dimension one which indicates its role as the direction of renormalization group flow.

The spinning D3 probe experiences the metric through the abelian Dirac-Born-Infeld action

$$S = -\tau_3 \int d^4\xi e^{-\phi} \sqrt{\det\left[G_{ab} + 2\pi\alpha' F_{ab}\right]} + \mu_3 \int C_4 \,, \tag{2.4}$$

where $\tau_3 = g_s^{-1}\mu_3$ are the D3 brane tension and charge respectively, and G_{ab} is the pullback of the metric[2]

$$G^{ab} = G_{MN} \frac{\partial x^M}{\partial \xi^a} \frac{\partial x^N}{\partial \xi^b} \,. \tag{2.5}$$

[2]We use $M,N = 0,\ldots,9$ for the $10d$ indices, $a,b = 0,\ldots,3$ for the world volume coordinates of the D3 brane and $m,n = 4,\ldots,9$ for the coordinates transverse to the brane.

Switching on only the scalar fields ($x^m = 2\pi\alpha' X_m = 2\pi\alpha'\varphi_m$) and allowing slow rotation on S^5 we find

$$L_{\rm probe} = \frac{\mu_3}{2} r^2 \dot{\Omega}_5^2 \,. \tag{2.6}$$

There is only a kinetic term so the position of the D3 brane (scalar vev) runs away to infinity as the field theory analysis predicted.

To stabilize the scalar sector we shall concentrate on a metastable vacuum where we allow the adjoint scalar fields to take non-commuting values. The scalar potential is of the form

$$V = \sum_{i=1}^{6} {\rm Tr} \left[\varphi_i^\dagger, \varphi_i\right]^2 \,, \tag{2.7}$$

and clearly contributes a quartic term when the vevs are non-commuting which will stabilize the negative mass term from the chemical potential. The resulting scalar vev will be of order μ. We perform an analysis of this vacuum in the next section in the D3 brane world volume theory.

Putting aside the scalar sector for the moment we can discuss what we would expect to happen in the fermionic sector were it in isolation. The dynamics of the theory should lie close to the Fermi surface so it is natural to look at an effective theory close to the Fermi surface following the analysis of [7, 9]. We write the fermions' momenta as a piece on the Fermi surface plus a small contribution perpendicular $\vec{p} = \vec{k} + \vec{l}$. Now as we scale towards the Fermi surface such that $l \to sl$ and $E \to sE$, with the scaling factor $s < 1$, for the fermion kinetic term to be marginal

$$\int dt d^3x \bar{\lambda} \partial^\mu \gamma_\mu \lambda \,, \tag{2.8}$$

we require $\lambda \to s^{-1/2}\lambda$. Under these scalings a four fermion interaction

$$G \int dt d^3p_1 d^3p_2 d^3p_3 d^3p_4 \bar{\lambda}\lambda\bar{\lambda}\lambda \delta(\vec{p_1} + \vec{p_2} - \vec{p_3} - \vec{p_4}) \tag{2.9}$$

is naively irrelevant (scaling as s) since the delta function is usually independent of $\vec{l}$ which are small. Only for scatterings between two fermions with equal and opposite momenta do the factors of $\vec{k}$ cancel in the delta function leaving it dependent on $\vec{l}$ and hence the operator marginal. Close to the Fermi surface there is therefore a vast simplification of the theory to these scatterings with special kinematics. In fact this has the effect of simplifying loop diagrams in the same way as the large N expansion in a theory with four fermion interactions. The running of the marginal operator is just through the "bubble sum" diagrams and results in a logarithmic running with an IR pole (whose position is dependent on the initial strength of the coupling) where Cooper pairing is expected to occur from the resulting strong interaction. In fact the gap equation truncation of the Schwinger Dyson equation becomes exact and the theory and its condensate are exactly soluble. Obviously this four fermion theory analysis is naive for a gauge theory but it is indicative of the expected behaviour in the presence of an effective four fermion vertex from gluon exchange.

The preferred condensate turns out [8, 9] to be between two like helicity spinors or, since they have opposite momentum, the anti-symmetric spin singlet state. Not surprisingly the preferred colour state is simply that with the most attractive interaction before the running. In the case of $\mathcal{N} = 4$ SYM the most attractive channel between adjoint gauginos is the symmetric colour singlet. The flavour structure is then determined by Fermi Dirac statistics to be a symmetric state. In other words we expect the condensate to be an element of the $4 \times 4 = 10$ dimensional representation of $\mathrm{SU}(4)_R$.

3. The spinning fuzzy D5 brane

In [10] the solution for a spinning D2 brane with surface D0 brane charge in flat spacetime was presented. In fact the construction is equally valid for D3 branes non-commutatively blowing up into a spinning D5, so we will review the construction in [10] applied to the D3 brane case which is relevant to our problem. We will assume the D3 branes are flat in the directions 0123 and study their motion in the six transverse directions. The effective action for N coincident D3 branes is given by [13]

$$S_{BI} = -\tau_3 \int d^4x Tr \left(e^{-\phi} \sqrt{-det(P[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij} E_{jb}] + \lambda F_{ab}) \det(Q^i_j)} \right), \tag{3.1}$$

where P indicates the pull back, $E_{ab} = G_{ab} + B_{ab}$, $\lambda = 2\pi\alpha'$, and

$$Q^i_j = \delta^i_j + i\lambda [X^i, X^k] E_{kj} \,. \tag{3.2}$$

The second term in the first determinant is zero when the metric is diagonal.

The resulting action for the scalar fields in Minkowski space, describing motion in the 6 transverse dimensions ($i = 4, \ldots, 9$), is given by

$$S = \tau_3 \int d^4x Tr \left(\frac{1}{2} \dot{X}^i \dot{X}^i + \frac{\lambda}{4} [X^i, X^j][X^i, X^j] \right). \tag{3.3}$$

The essence of the construction is to use the positive potential from the non-commutativity to provide the centrifugal force to sustain rotational motion. It is not possible to support a 2-sphere by rotation within the world volume of the 2-sphere. Instead we must allow the 2-sphere to topple in the $6d$ space. We embed the 2-sphere in the directions 468. Then we pair the three axes each with an axis in the transverse $3d$ space 579 and allow rotation in each of the three resulting planes. The ansatz for the N D3 to have non-commutatively puffed up and to be spinning in the three separate transverse planes (45, 67, 89) with the same angular velocity ω is [10]

$$\begin{aligned}
X_4(t) &= \frac{2}{\sqrt{N^2-1}} T^1 r_4(t) \ , \ X_5(t) = \frac{2}{\sqrt{N^2-1}} T^1 r_5(t), \\
X_6(t) &= \frac{2}{\sqrt{N^2-1}} T^2 r_6(t) \ , \ X_7(t) = \frac{2}{\sqrt{N^2-1}} T^2 r_7(t), \\
X_8(t) &= \frac{2}{\sqrt{N^2-1}} T^3 r_8(t) \ , \ X_9(t) = \frac{2}{\sqrt{N^2-1}} T^3 r_9(t) \, ,
\end{aligned} \tag{3.4}$$

where the T^i are the three generators of SU(2) in the $N \times N$ irreducible representation. They have the properties

$$[T^i, T^j] = i\epsilon_{ijk} T_k \, , \qquad \sum_i T_i^2 = \frac{N^2-1}{4}\mathbf{I} \, , \qquad Tr(T_i^2) = \frac{N(N^2-1)}{12} \, . \tag{3.5}$$

Substituting into the action, using a Minkowski metric, produces a lagrangian

$$\begin{aligned}
L = \frac{N\tau_3}{3} \Bigg(\frac{1}{2} \sum_{i=4}^{9} \dot{r}_i^2 - & \qquad (3.6) \\
- \frac{\alpha^2}{2} \left[(r_4^2 + r_5^2)(r_6^2 + r_7^2) + (r_4^2 + r_5^2)(r_8^2 + r_9^2) + (r_6^2 + r_7^2)(r_8^2 + r_9^2) \right] \Bigg) \, ,
\end{aligned}$$

where $\alpha = 1/(\pi l_s^2 \sqrt{N^2-1})$. The equations of motion are thence

$$\begin{aligned}
\ddot{r}_{4(5)} &= -\alpha^2 (r_6^2 + r_7^2 + r_8^2 + r_9^2) r_{4(5)} \\
\ddot{r}_{6(7)} &= -\alpha^2 (r_4^2 + r_5^2 + r_8^2 + r_9^2) r_{6(7)} \\
\ddot{r}_{8(9)} &= -\alpha^2 (r_6^2 + r_7^2 + r_4^2 + r_5^2) r_{8(9)} \, ,
\end{aligned} \tag{3.7}$$

which have solutions

$$\begin{aligned}
r_4 &= R\cos\omega t \, , & r_5 &= R\sin\omega t \, , \\
r_6 &= R\cos\omega t \, , & r_7 &= R\sin\omega t \, , \\
r_8 &= R\cos\omega t, & r_9 &= R\sin\omega t \, ,
\end{aligned} \tag{3.8}$$

with

$$\omega = \sqrt{2}\alpha R \, . \tag{3.9}$$

We have then a stable solution of the form we seek. Ideally we should now find the background geometry in the large N limit and take the near horizon limit to determine the dual field theory behaviour. However, this is clearly a huge task since the construction has time dependent motion and couples to the metric, the RR 6-form and RR 4-form potentials. It is not even clear a priori that such a solution would exist as a self consistent solution although it is likely. Instead we will retreat to an easier problem that will nevertheless display the important physics. Instead of spinning all N D3 branes we will choose instead to spin $n \ll N$ corresponding to introducing a chemical potential for only n of the gauge degrees of freedom. In terms of the D3 computation we will therefore be able to work in an n/N expansion. Essentially we are including a probe D5 brane with surface D3 charge. To leading order the background metric is that of $AdS_5 \times S^5$ in (2.3). Inserting the background into the non-abelian Born Infeld action we find

$$S_{BI} = -\tau_3 \int d^4x Tr \sqrt{H^{-2}(1 + H\dot{X}_i)}(1 - \frac{\lambda^2}{4} H[X^i, X^j]^2) + \tau_3 \int d^4x H^{-1} . \quad (3.10)$$

The potential term vanishes as usual and in the remaining leading terms the factors of H cancel leaving precisely the Minkowski space action we had before. This cancellation is to be expected since the action should describe the $\mathcal{N} = 4$ Yang Mills theory where H is absent. The computation in AdS, therefore, exactly mirrors [10] and the construction is seen to also exists in AdS. We will now go on to analyze the effects of the construction on the background supergravity solution to see if a fermion condensate is indeed present.

4. Asymptotic operators

Having determined that there is a meta-stable vacuum where the D3 branes balance their rotational motion against non-commutative expansion, we wish to ask what operators in addition to the chemical potential characterize the vacuum. There is a D5 source in the interior of the space which will give rise to a non-zero 3-form. To see this we look at the supergravity equations of motion linearized around the $AdS_5 \times S^5$ background. At linear order in the n/N expansion we can neglect the back-reaction of the brane on the metric and the RR 4-form, and simply consider the linearized equation for a 3-form field

$$G_3 = F_3 - \hat{\tau} H_3 . \quad (4.1)$$

Here F_3 ,H_3 are the RR and NS 3-form, and $\hat{\tau} = i/g$ is the background value of the dilaton field. The linearized equation of motion and Bianchi identity for the field G_3 were shown by Polchinksi and Strassler [4] to take the simple form[3]

$$\begin{aligned} d(r^4(*_6 G_3 - iG_3)) &= 0 , \\ dG_3 &= J_4 , \end{aligned} \quad (4.2)$$

where $*_6$ indicates dualizing in the the six dimensional transverse space using a flat metric for contractions, and J_4 is the D5 brane source.

Consider first a static D5 brane lying in R^4, and wrapped as a 2-sphere in the transverse six dimensional space [4]. We call the three coordinates in which the

[3]In the following we set the AdS radius, L, to one.

2-sphere lies the w directions and it is then at the origin in the remaining three y directions. It acts as an electric source for the 6-form potential with components in the $0 \ldots 4, \theta_w, \phi_w$ directions. Dualizing to find the magnetic source for the 3-form gives

$$J_4 = 4\pi^2 \alpha' \delta^3(y) \delta(w - r_0) d^3 y \wedge dw \,, \tag{4.3}$$

where w is the radial coordinate on the sphere and r_0 is the radius of the sphere which we saw above is proportional to the angular velocity.

It is convenient to write G_3 in terms of a potential

$$G_3 = *_6 dw_2 + i dw_2 \,. \tag{4.4}$$

Working in the gauge $dw_2 = 0$, this ansatz solves the equation of motion leaving just the Bianchi identity with source

$$\partial_m \partial_m w_2 = \frac{2\pi^2 \alpha'}{r_0} \delta^3(y) \delta(w - r_0) \epsilon_{ijk} w^i dw^j \wedge dw^k \,, \tag{4.5}$$

which has the asymptotic solution [4]

$$w_2 \sim -\frac{8\alpha' r_0^3}{3r^6} \epsilon_{ijk} w^i dw^j \wedge dw^k \,. \tag{4.6}$$

The solution scales (in an inertial frame transverse to the radial direction) as $1/r^3$, the normalizable solution for an operator of dimension 3. This is the appropriate behaviour for the gaugino Cooper pair condensate we seek.

We must also check that the symmetry properties are correct. Again following Polchinski and Strassler [4] we adopt complex coordinates

$$z_1 = \frac{w^1 + iy^1}{\sqrt{2}} \,, \qquad z^2 = \frac{w^2 + iy^2}{\sqrt{2}} \,, \qquad z_3 = \frac{w^3 + iy^3}{\sqrt{2}} \,. \tag{4.7}$$

Under a rotation $z^i \to e^{i\phi_i} z_i$ the gauginos transform as

$$\begin{aligned} \lambda_1 &\to e^{i(\phi_1 - \phi_2 - \phi_3)/2} \lambda_1 \,, \\ \lambda_2 &\to e^{i(-\phi_1 + \phi_2 - \phi_3)/2} \lambda_2 \,, \\ \lambda_3 &\to e^{i(-\phi_1 - \phi_2 + \phi_3)/2} \lambda_3 \,, \\ \lambda_4 &\to e^{i(\phi_1 + \phi_2 + \phi_3)/2} \lambda_4 \,. \end{aligned} \tag{4.8}$$

Thus we can construct a 3-form with the same symmetry transformations as a condensate

$$\langle \lambda_1 \lambda_1 \rangle dz^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + \langle \lambda_2 \lambda_2 \rangle d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 + \langle \lambda_3 \lambda_3 \rangle d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^3 + \langle \lambda_4 \lambda_4 \rangle dz^1 \wedge dz^2 \wedge dz^3 \,. \tag{4.9}$$

When all four condensates are equal the 3-form, written in the real coordinates, becomes

$$\langle\lambda\lambda\rangle(dw^1 \wedge dw^2 \wedge dw^3 + i dy^1 \wedge dy^2 \wedge dy^3)\,. \tag{4.10}$$

We can now compare this to the solution we have for the 3-form field strength. The leading term has an epsilon tensor in the three w directions and hence has the correct symmetry properties to correspond to a real condensate of this form. This is essentially the result we were looking for — the introduction of a chemical potential, in a vacuum where the scalar instability has been removed, results in fermionic Cooper pair formation. The precise form of the condensate is though somewhat surprising. We introduced a chemical potential for the fermions of the form $diag(1,1,1,-3)$ and hence would have expected only an SU(3) symmetry in the condensate rather than the SU(4) symmetry observed. We note that the same lifting of the symmetry was observed in the $\mathcal{N}=1^*$ theory [4] — there the $\mathrm{SU}(4)_R$ symmetry was broken by a mass term for three of the four fermions. As noted there the interpretation is clouded by a three scalar operator also in the 10 representation $(6\times 6\times 6 = 10+\cdots)$ of $\mathrm{SU}(4)_R$ which can mix with the fermion condensate. In fact it makes no sense to distinguish these operators in the theory but it is nevertheless encouraging that such an operator is present.

In this analysis we have neglected the spin of the D5 which, however, is easily included. Equation (4.5) becomes the wave equation

$$(\partial_m^2 - \partial_t^2)w_2 = \frac{2\pi^2\alpha'}{r_0}\delta^3(Im(ze^{i\omega t}))\delta(Re(ze^{i\omega t}) - r_0)\epsilon_{ijk}z^i dz^j \wedge dz^k e^{i\omega t}\,, \tag{4.11}$$

where z are the complex coordinates. The asymptotic solution is

$$w_2 \sim -\frac{8\alpha' r_0^3}{3} z^i \frac{J_3(\omega r)}{r^3} e^{i\omega t}\,, \tag{4.12}$$

where J_3 is one of the Bessel functions of the second kind (we choose this solution since this Bessel function diverges in the interior). Expanding the Bessel function for small ω gives

$$w_2 \sim -\frac{8\alpha' r_0^3}{3} z^i \frac{F(\omega r)}{r^6} e^{i\omega t}\,, \tag{4.13}$$

where $F(\omega r) = 1 + \mathcal{O}(\omega^2 r^2)$ is an oscillatory function in r.

The result still describes a condensate except that there is a sinusoidal oscillation between the w and y directions. In terms of the condensate this implies a time dependent shift in the global phase. The spinning configuration is in fact giving off waves of C_2 potential precisely of the type found in [10] in Minkowski space. The angular momentum ($\mathrm{U}(1)_R$ charge in the field theory) is radiating away from the configuration and out through the AdS space. A full analysis would also reveal C_4 and gravitational radiation. Over time this radiation will spin the construction

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down to a static state with the angular momentum lost to the RR potentials radiating away. What is the interpretation of this phenomena in terms of the dual field theory?

We know of course that the construction is only a metastable vacuum with the true vacuum being a runaway in the scalar vev. We must therefore expect to see the construction decay. The decay can occur either through tunnelling or via higher dimension couplings between the scalars in the D5 brane's world volume and other unbounded operators in the theory. These operators are precisely what the couplings to the RR forms represent. The RR form vevs are sources describing the full set of primary operators in the theory so the radiation indeed describes the decay of the metastable vacuum to runaway operators. As in the case where the angular momentum is endowed to commutative D3 branes the signal of the runaway is that the angular momentum is carried to the edge of the moduli space - in that case by the D3 brane motion whilst in this more complicated case by the RR forms.

5. Discussion

We have shown that a fuzzy D5 sphere may be supported in the AdS/CFT correspondence by rotation on the S^5 sphere of a small number of the D3 branes. In the field theory the rotation corresponds to the inclusion of a chemical potential putting the theory at high density with respect to a $\mathrm{U}(1)_R$ symmetry group. The $\mathcal{N} = 4$ theory is unbounded in the scalar sector by a chemical potential but this state represents a metastable vacuum where the negative mass is balanced against a positive potential from non-commutativity of the scalar fields. The D5 brane necessarily couples to a C_6 (or dual C_2) potential the form of which we have shown corresponds to an operator in the field theory with the dimension and symmetry properties of a fermion condensate. This matches our expectation that at high density Cooper pair formation should result from the attractive gauge interactions. In the future it remains as a challenge to establish the existence of such a configuration when all the D3 branes are rotating and find the full supergravity background. It would also be nice to study the phenomena in a theory, with a gravity dual, where there are no scalar fields (such as $\mathcal{N} = 1$ Yang Mills theory) since in the $\mathcal{N} = 4$ theory scalar operators mix with the fermion condensate muddying the interpretation.

The rotating D5 brane is in fact a time dependent structure since it topples in the S^5 space. This results, through its couplings to RR-forms, in its decay by the emission of waves. We have interpreted this as decay of the metastable vacuum to the true unbounded vacuum of the $\mathcal{N} = 4$ theory at high density, though it would be interesting to further study the operators present at asymptotic r in (4.12) which presumably describe the operators of the runaway vacuum.

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